Coordinated graphs and clique graphs of clique-Helly perfect graphs

Flavia Bonomo[1]

Dep. de Computación, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Buenos Aires, Argentina. fbonomo@dc.uba.ar

Guillermo Durán[2]

Dep. de Ingeniería Industrial, Facultad de Ciencias Físicas y Matemáticas, Universidad de Chile, Santiago, Chile. Fax number: (56) (2) 689-7895. gduran@dii.uchile.cl

Marina Groshaus[1]

Dep. de Computación, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Buenos Aires, Argentina. groshaus@dc.uba.ar

Abstract

A new class of graphs related to perfect graphs is defined in this work: coordinated graphs. A graph G is coordinated if the cardinality of a maximum set of cliques of H with a common vertex is equal to the cardinality of a minimum partition of the cliques of H into clique-independent sets, for every induced subgraph H of G. A graph G is K-perfect when its clique graph $K(G)$ is perfect. The concept of *special clique subgraph* is defined, which leads us to the notion of c–coordinated graphs (coordination relative to these clique subgraphs). We prove that coordinated graphs are a subclass of perfect graphs and relate K-perfect graphs with c–coordinated graphs. Finally, clique graphs of clique-Helly and hereditary clique-Helly perfect graphs are analyzed.

Key words: clique graphs, clique-Helly graphs, coordinated graphs, hereditary clique-Helly graphs, K-perfect graphs, perfect graphs.

1 Introduction

Let G be a finite undirected graph, $V(G)$ and $E(G)$ the vertex and edge sets of G, respectively. Denote $|V(G)| = n$. A clique in a graph is a complete subgraph maximal under inclusion. A stable set in a graph is a subset of pairwise non-adjacent vertices of it. The stability number $\alpha(G)$ is the cardinality of a maximum stable set of G. The neighbourhood of a vertex v is the set $N(v)$ consisting of all the vertices which are adjacent to v. The closed neighbourhood of v is $N[v] = N(v) \cup \{v\}$.

Let $C(G)$ be the set of cliques of G. Denote $|C(G)| = k$. Let v and w be vertices of G. Let $C((v,w))$ and $C(v)$ be the sets of cliques containing the edge (v,w) and the vertex v, respectively. Vertices which belong to exactly one

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clique will be called simplicial vertices. A vertex v dominates a vertex w in G if $C(w) \subseteq C(v)$. Two vertices v and w are *twins* if $C(v) = C(w)$.

The chromatic number of a graph G is the smallest number of colors that can be assigned to the vertices of G in such a way that no two adjacent vertices receive the same color, and is denoted by $\chi(G)$. An obvious lower bound is the maximum cardinality of the cliques of G, the clique number of G, denoted by $\omega(G)$.

Berge [2] proposed to call a graph G perfect whenever $\chi(H) = \omega(H)$ for every induced subgraph H of G. Perfect graphs are very interesting from an algorithmic point of view. While determining the clique number and the chromatic number of a graph are NP-complete problems, they are solvable in polynomial time for perfect graphs [14]. Besides, it has been proved recently that perfect graphs can be characterized by forbidden subgraphs [6] and recognized in polynomial time [7]. For more background information on perfect graphs see [13].

A clique-transversal of a graph G is a subset of vertices that meets all the cliques of G. A clique-independent set is a collection of pairwise vertex-disjoint cliques. The clique-transversal number and clique-independence number of G, denoted by $\tau_C(G)$ and $\alpha_C(G)$, are the sizes of a minimum clique-transversal and a maximum clique-independent set of G, respectively. It is easy to see that $\tau_C(G) \geq \alpha_C(G)$ for any graph G. As it is defined in [15], a graph G is clique-perfect if $\tau_C(H) = \alpha_C(H)$ for every induced subgraph H of G.

A family of subsets S satisfies the Helly property when every subfamily of it consisting of pairwise intersecting subsets has a common element. A graph is clique-Helly (CH) when its cliques satisfy the Helly property. A graph G is hereditary clique-Helly (HCH) when H is clique-Helly for every induced subgraph H of G. Both classes of graphs can be recognized in polynomial time [25, 21]. An interesting survey on clique-Helly and hereditary clique-Helly graphs appears in [12].

Let $M_1, \ldots, M_k$ and $v_1, \ldots, v_n$ be the cliques and vertices of a graph G, respectively. A clique matrix $A_G \in R^{k \times n}$ of G is a 0-1 matrix whose entry a_{ij} is 1 if $v_j \in M_i$, and 0, otherwise.

Consider a finite family of non-empty sets. The intersection graph of this family is obtained by representing each set by a vertex, two vertices being connected by an edge if and only if the corresponding sets intersect. The clique graph $K(G)$ of G is the intersection graph of the cliques of G. The graph $K^2(G)$ is the clique graph of $K(G)$. A graph G is K-perfect if its clique graph $K(G)$ is perfect.

Let $A \in R^{r \times n}$ be a 0-1 matrix having no zero columns. The derived graph of A is the intersection graph of its columns, that is, a graph of n vertices $v_1, \ldots, v_n$ where v_i is adjacent to v_j if there exists a row l in A such that $a_{li} = a_{lj} = 1$.

We define a new class of graphs related to perfect graphs: coordinated graphs. Let v be a vertex of a graph G. Denote $m(v) = |C(v)|$. Let $M(G)$ be the maximum $m(v)$ for any v in G. Let $F(G)$ be the cardinality of a minimum partition of the cliques of G into clique-independent sets, that is, the smallest number of colors that can be assigned to the cliques of G so that intersecting cliques have different colors. Note that $F(G) \geq M(G)$ for any graph G. We say that a graph G is coordinated if $F(H) = M(H)$, for every induced subgraph H of G.

The concept of *special clique subgraph* is defined in this work, which leads us to the notion of coordination relative to these clique subgraphs. A graph G is c–coordinated if $F(H) = M(H)$ for every special clique subgraph H of G.

Let $\mathcal{H}$ be a class of graphs and $K(\mathcal{H})$ be the class of clique graphs of graphs in $\mathcal{H}$. If $K(\mathcal{H}) = \mathcal{H}$, we say that $\mathcal{H}$ is fixed under the clique operator K. Clique-Helly and hereditary clique-Helly graphs are classes with this property [11, 21]. Two classes of graphs $\mathcal{H}$ and $\mathcal{L}$ are dual-clique classes if $K(\mathcal{H}) = \mathcal{L}$ and $K(\mathcal{L}) = \mathcal{H}$. Examples of dual-clique graph classes appear in several works (see [3, 4, 5, 22, 26]).

A characterization for the class of clique graphs was formulated by Roberts and Spencer [23], inspired by a paper of Hamelink [16], but no efficient algorithm is known based on this characterization. In fact, it is an open problem whether or not the problem of recognizing clique graphs is NP-complete. Clique graphs of several classes of graphs have already been studied in the literature: trees [17], interval graphs [18], Helly circular-arc graphs [9], disk-Helly graphs [1], chordal graphs [26], are some of them. An interesting survey on clique graphs can be found in [24].

In this paper, we prove that coordinated graphs are a subclass of perfect graphs. Furthermore, a characterization of HCH K-perfect graphs using clique subgraphs is formulated. Finally, clique graphs of clique-Helly and hereditary clique-Helly perfect graphs are studied. We prove that (hereditary) clique-Helly perfect graphs and (hereditary) clique-Helly K-perfect graphs are dual-clique classes of graphs.

2 Coordinated graphs

Coordinated graphs are perfect graphs. In order to prove this, we need some previous results.

Clearly, C_{2r+1} is not coordinated for $r \geq 2$, because $M(C_{2r+1}) = 2$ while $F(C_{2r+1}) = 3$.

Let $\overline{C_n}$, with $n \geq 5$, be the complement of an induced cycle $v_1, \ldots, v_n$, that is, v_i and v_j are adjacent if and only if $j \neq i-1, i+1$ (from now on, all the indices must be understood modulo n).

We will prove that $\overline{C_n}$ is not coordinated for $n \geq 5$, $n \neq 6$. We need some results related to $M(\overline{C_n})$ and $|C(\overline{C_n})|$ in order to conclude that $F(\overline{C_n}) > M(\overline{C_n})$ for $n \geq 5$, $n \neq 6$.

Let A_n be the number of sequences $[a_1, \ldots, a_s]$ where $a_i \in \{2, 3\}$ and $\sum_{i=1}^{s} a_i = n$.

Lemma 2.1 *In $\overline{C_n}$, there is a one-to-one correspondence between the cliques in the set $C(v_i)$ and the sequences $[a_1, \ldots, a_s]$ such that $a_j \in \{2, 3\}$ and $\sum_{j=1}^{s} a_j = n$. Therefore $|C(v_i)| = A_n$.*

Proof: Without loss of generality, suppose $i = 1$. Let $D \in C(v_1)$, $D = \{v_{i_1}, \ldots, v_{i_s}\}$, $1 = i_1 < \cdots < i_s$. Given two consecutive vertices v_i, v_{i+1} in $\overline{C_n}$, every clique contains at most one of them. So $i_{j+1} - i_j \geq 2$ and $i_s \leq n-1$. On the other hand, by maximality, given three consecutive vertices v_i, v_{i+1} and v_{i+2} in $\overline{C_n}$, every clique contains at least one of them. So $i_{j+1} - i_j \leq 3$ and $i_s \geq n-2$. Then, we can assign to D the sequence

$S(D) = [i_2 - i_1, \ldots, i_s - i_{s-1}, n+1-i_s]$, where every element is equal to 2 or 3 and its sum is $(i_2 - i_1) + \cdots + (i_s - i_{s-1}) + (n+1-i_s) = n+1-i_1 = n$.

Conversely, let $S = [a_1, \ldots, a_s]$ be a sequence such that $a_j \in \{2,3\}$ and $\sum_{j=1}^{s} a_j = n$. We can assign to this sequence the set of vertices $D(S) = \{v_{i_1}, \ldots, v_{i_s}\}$ where $i_1 = 1$ and $i_j = i_{j-1} + a_{j-1}$ for $j = 2, \ldots, s$. To verify that $D(S)$ is a clique, observe that since $1 = i_1 < \cdots < i_s$, $i_{j+1} - i_j = 2$ or 3, and $i_s \leq n+1-2 = n-1$, there are not two consecutive vertices in $D(S)$. Therefore, $D(S)$ is complete. Let v_i be a vertex different from $v_{i_1}, \ldots, v_{i_s}$. Then, there is an index j such that $i_j < i < i_{j+1}$, or $i_s < i \leq n$. In the first case, as $i_{j+1} - i_j = 2$ or 3, it follows that $i = i_{j+1} - 1$ or $i = i_j + 1$. In the second case, as $i_s \geq n+1-3 = n-2$, then $i = i_s + 1$ or $i = n$. In both cases, $D(S) \cup \{v_i\}$ does not induce a complete subgraph. Then, $D(S)$ induces a complete maximal subgraph.

Finally, observe that this correspondences are dual, that means $D(S(D)) = D$ and $S(D(S)) = S$. Therefore, $|C(v_i)| = A_n$. ∎

Lemma 2.2 *In $\overline{C_n}$, there is a one-to-one correspondence between the cliques in the set $C((v_i, v_{i+3}))$ and the sequences $[a_1, \ldots, a_s]$ such that $a_j \in \{2,3\}$ and $\sum_{j=1}^{s} a_j = n-3$. As a consequence, $|C_{(v_i, v_{i+3})}| = A_{n-3}$.*

Proof: Without loss of generality, we can assume that $i = 1$. Consider the assignment of sequences in the proof of Lemma 2.1. Let $D \in C(v_1)$. Then $D \in C((v_1, v_4))$ if and only if $a_1 = 3$ in $S(D)$, otherwise $a_1 = 2$ and $v_3 \in D$, which would imply that $v_4 \notin D$, which is a contradiction. Then, there is a one-to-one correspondence between the cliques in the set $C((v_i, v_{i+3}))$ and the sequences $[a_1, \ldots, a_s]$ such that $a_j \in \{2,3\}$, $a_1 = 3$ and $\sum_{j=1}^{s} a_j = n$, or equivalently, the sequences $[a_1, \ldots, a_t]$ such that $a_j \in \{2,3\}$ and $\sum_{j=1}^{t} a_j = n-3$. Therefore, $|C_{(v_i, v_{i+3})}| = A_{n-3}$. ∎

Theorem 2.1 *$M(\overline{C_n}) = A_n$ and $|C(\overline{C_n})| = 2A_n + A_{n-3}$.*

Proof: By lemma 2.1, for every $i = 1, \ldots, n$ it holds that $m_{\overline{C_n}}(v_i) = |C(v_i)| = A_n$. Then $M(\overline{C_n}) = A_n$. Since the set of cliques of $\overline{C_n}$ is the disjoint union of $C((v_i, v_{i+3}))$, $C(v_{i+1})$ and $C(v_{i+2})$, the fact that $|C(\overline{C_n})| = 2A_n + A_{n-3}$ follows directly from Lemma 2.1 and Lemma 2.2. ∎

Clearly, the following lemma holds.

Lemma 2.3 *Let $\mathcal{P}$ be a partition of the cliques of $\overline{C_n}$ into clique-independent sets. Every clique of $C(v_i)$ whose assigned sequence has $a_j = 2$ for some j, belongs to a set of cardinality at most 2 in $\mathcal{P}$.*

Using this facts, this theorem can be proved.

Theorem 2.2 *The graph $\overline{C_n}$ is not coordinated for $n \geq 5$, $n \neq 6$.*

Proof: Let us see that for $n \geq 5, n \neq 6$, $F(\overline{C_n}) > M(\overline{C_n})$. Let $\mathcal{P}$ be a minimum partition of the cliques of $\overline{C_n}$ into clique-independent sets. Let D be a clique in $C(v_i)$ and $[a_1, \ldots, a_s]$ be the sequence associated to D given by Lemma 2.1.

If n is not a multiple of 3, there must be some index j such that $a_j = 2$. As we observed before, the subset of the partition $\mathcal{P}$ that contains the clique D has at most one more clique. Since D is an arbitrary clique, we conclude that the cardinality of every set in the partition is at most 2. Then, it holds:

$$F(\overline{C_n}) = |\mathcal{P}| \geq \frac{|C(\overline{C_n})|}{2} = \frac{2A_n + A_{n-3}}{2} > A_n = M(\overline{C_n}),$$

for $n \geq 5$, $A_{n-3} > 0$.

If $n = 3t$, where $t \geq 3$, then there are exactly three cliques in $\overline{C_n}$ that can be represented with sequences such that $a_j = 3$ for every j: $M_1 = \{v_1, v_4, \ldots, v_{n-2}\}$, $M_2 = \{v_2, v_5, \ldots, v_{n-1}\}$ and $M_3 = \{v_3, v_6, \ldots, v_n\}$. This means that there is at most one set of three independent cliques in the partition $\mathcal{P}$, and the cardinality of the other sets must be at most two. Using the fact that $A_{3t-3} > 1$ for $t \geq 3$, we obtain:

$$F(\overline{C_{3t}}) = |\mathcal{P}| \geq \frac{|C(\overline{C_{3t}})| - 1}{2} = \frac{2A_{3t} + A_{3t-3} - 1}{2} > A_{3t} = M(\overline{C_{3t}})$$

■

Corollary 2.1 *Coordinated graphs are perfect graphs.*

Proof: It is a direct consequence of the fact that C_{2r+1} is not coordinated for $r \geq 2$, Theorem 2.2, and the Strong Perfect Graph Theorem [6], which claims that a graph is perfect if and only if it contains neither an induced odd cycle of length at least five nor its complement. ■

From the proof of Theorem 2.2 it follows that, for $n \geq 5$,

$$F(\overline{C_n}) - M(\overline{C_n}) \geq \frac{A_{n-3} - 1}{2},$$

and since $\{A_n\}_{n \geq 0}$ grows in an exponential way, the family $\{\overline{C_n}\}_{n \geq 7}$ turns to be a family of highly non-coordinated graphs (the difference between $F(G)$ and $M(G)$ can be arbitrarily large).

In this sense, this family is similar to the family of highly imperfect graphs presented by Mycielski [20], and to the family of highly clique-imperfect graphs presented in [10], where the differences between $\chi(G)$ and $\omega(G)$, and between $\tau_C(G)$ and $\alpha_C(G)$ respectively, can be arbitrarily large.

3 K-perfect graphs

Coordinated graphs and K-perfect graphs are related by the following theorem.

Theorem 3.1 *Let G be a graph. Then:*

(i) $F(G) = \chi(K(G))$.

(ii) $M(G) \leq \omega(K(G))$.

(iii) *If G is clique-Helly then $M(G) = \omega(K(G))$.*

Proof:

(i) Let $F_1, \ldots, F_{F(G)}$ be a partition of the cliques of G into clique-independent sets. This partition induces a partition of the vertices of $K(G)$ into stable sets, which gives a coloring for $K(G)$. Then, $\chi(K(G)) \leq F(G)$. Analogously, let $F_1^{'}, \ldots, F_{\chi(K(G))}^{'}$ be the partition of the vertices of $K(G)$ into stable sets induced by an optimal coloring of $K(G)$. Considering the vertices of $K(G)$ as cliques in G, we obtain a partition of the cliques of G into clique-independent sets. Then $F(G) \leq \chi(K(G))$.

(ii) Observe that $m(v) \leq \omega(K(G))$, $\forall v \in V(G)$, since all the vertices that correspond to the $m(v)$ cliques containing v induce a complete subgraph in $K(G)$. In particular, $M(G) \leq \omega(K(G))$.

(iii) We only need to prove that if G is clique-Helly, then $\omega(K(G)) \leq M(G)$. Let L be a maximum clique of $K(G)$ and $M_1, \ldots, M_r$ be the cliques of G that correspond to the vertices of L. Since G is a clique-Helly graph, there is at least one vertex v_L in G which belongs to the intersection of all the r cliques. So, it is easy to see that $M(G) \geq m(v_L) = \omega(K(G))$. ∎

Let G be a graph, $\{M_1, \ldots, M_k\}$ be the cliques of G, and $\{i_1, \ldots, i_s\}$ be a subset of $\{1, \ldots, k\}$. The graph $G_{i_1,\ldots,i_s}$, formed by the vertices and edges of $M_{i_1}, \ldots, M_{i_s}$, is a clique subgraph of G. We say that $G_{i_1,\ldots,i_s}$ is a *special clique subgraph* of G if all the cliques of $G_{i_1,\ldots,i_s}$ are cliques of G, and that the graph G is *cliqual* if, for every subset $\{i_1, \ldots, i_s\}$ of $\{1, \ldots, k\}$, the cliques of $G_{i_1,\ldots,i_s}$ are exactly $M_{i_1}, \ldots, M_{i_s}$. Clearly, if the graph G is cliqual, every clique subgraph is special.

Theorem 3.2 *Let G be a graph. Every clique subgraph of G is an induced subgraph of G if and only if G does not have either P_4 or C_4 as induced subgraph* (Figure 1).

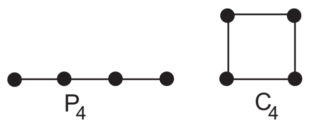

Figure 1: P_4 and C_4.

Proof: $\Rightarrow$) Suppose that either P_4 or C_4 is an induced subgraph of G. As we can observe in Figure 2, the edge (v, w') belongs to a clique M_i and the edge (v', w) belongs to a clique M_j. Clearly, $G_{i,j}$ is a clique subgraph of G with the edge (v, w) missing. Then G has a clique subgraph which is not induced.

$\Leftarrow$) Suppose G has a clique subgraph H which is not induced. That means there exists an edge (v, w) which belongs to $E(G)$ but not to $E(H)$, with v and w belonging to $V(H)$. Then, there must be a clique M_i of G containing v but not w, and a clique M_j of G containing w but not v.

As the vertex w does not belong to M_i but is adjacent to v in G, there must be a vertex w' in M_i such that (w, w') is not in $E(G)$. Analogously, there exists a vertex v' in M_j, with (v, v') not in $E(G)$. Finally, either $(v', w') \in E(G)$ and v', w, v, w' induce the graph C_4, or v', w, v, w' induce the graph P_4. ∎

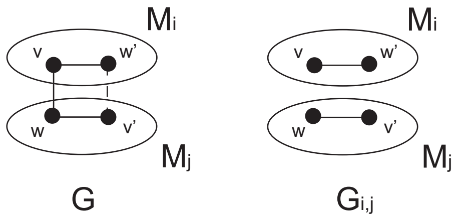

Figure 2: A clique subgraph of G which is not induced.

Remark 3.1 *Graphs which do not have either P_4 or C_4 as induced subgraphs are called trivially perfect graphs* [13].

Our next result relates hereditary clique-Helly graphs to cliqual graphs. The following characterization of hereditary clique-Helly graphs given by Prisner is needed [21].

Theorem 3.3 *A graph G is hereditary clique-Helly if and only if it does not contain the following graphs as induced subgraphs:*

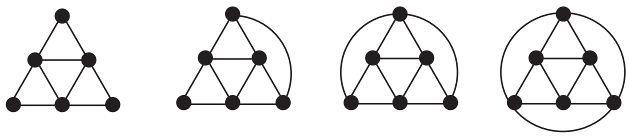

Figure 3: Hajös graphs.

Theorem 3.4 *Let G be a graph. Then G is cliqual if and only if G is an hereditary clique-Helly graph.*

Proof: $\Rightarrow$) By Theorem 3.3, if G is not hereditary clique-Helly, it must contain any of the following graphs as an induced subgraph:

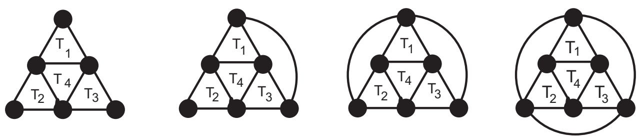

Figure 4: Forbidden subgraphs for hereditary clique-Helly graphs.

Then, there exist three cliques in G, M_{i_1}, M_{i_2} and M_{i_3}, containing triangles T_1, T_2 and T_3, respectively. It is clear that T_4 is included in none of them. We can see that the subgraph G_{i_1,i_2,i_3} contains T_4. This complete subgraph belongs to a clique in G_{i_1,i_2,i_3} which is different from M_{i_1}, M_{i_2} and M_{i_3}. Then G is not cliqual.

$\Leftarrow$) Suppose that G is not cliqual. This means that there exist a set of indices $\{i_1, \ldots, i_s\}$ such that the set of cliques of the subgraph $G_{i_1,\ldots,i_s}$ properly includes the cliques $M_{i_1}, \ldots, M_{i_s}$. Let $H = G_{i_1,\ldots,i_s}$ and $\mathcal{F} = \{M_{i_1}, \ldots, M_{i_s}\}$. Let M be a clique of H not belonging to $\mathcal{F}$ and $m = |M|$. Consider the set $A = \{j : 1 \leq j \leq m /\ \forall U \subseteq M,\ |U| = j,\ U \textit{ is covered by a clique of } \mathcal{F}\}$. Since every vertex of H belongs to some clique of $\mathcal{F}$, A is bounded and not empty. Let $r = max_j\{j \in A\}$. Observe that, as every subset of M with cardinality two is an edge of H contained in some clique of $\mathcal{F}$, $r \geq 2$. Also, it holds that $r < m$, otherwise the clique M would be covered by a clique of $\mathcal{F}$, which leads to a contradiction. Then there exist some subset $R \subseteq M$ of cardinality $r + 1$ which cannot be covered by any clique of $\mathcal{F}$. As $r + 1 \geq 3$, let u, v and w be three different vertices of R. As $r \in A$, $R - \{u\}$ is covered by a clique M_u of $\mathcal{F}$. Clearly, $u \notin M_u$, otherwise R would be covered by a clique of $\mathcal{F}$, which is a contradiction. This means that there is a vertex $u' \in M_u$ which is not adjacent to u in G. Analogously, $R - \{v\}$ is covered by a clique M_v of $\mathcal{F}$ such that $v \notin M_v$, $R - \{w\}$ is covered by a clique M_w of $\mathcal{F}$ with $w \notin M_w$, and there are two vertices $v' \in M_v$ and $w' \in M_w$ which are not adjacent in G to v and w, respectively.

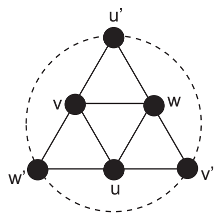

Figure 5: Scheme of adjacency relations between vertices u, v, w, u', v' and w'.

Finally, u, v, w, u', v' and w', depending on whether u', v' and w' are adjacent in G or not (as it can be seen in Figure 5), induce in G some of the forbidden subgraphs for hereditary clique-Helly graphs, according to Theorem 3.3. ∎

A property is clique-hereditary when, if it holds for G, it holds for every clique subgraph of G. Note that the clique-Helly property and the cliqual property are clique-hereditary. Being equivalent to the hereditary clique-Helly property, the cliqual property is also hereditary.

Special clique subgraphs of G and induced subgraphs of $K(G)$ are related. The following theorem enables to relate c–coordinated graphs to K-perfect graphs.

Theorem 3.5 *Let G be a graph:*

(i) *If H is a special clique subgraph of G, then $K(H)$ is an induced subgraph of $K(G)$.*

(ii) *If G is an hereditary clique-Helly graph, then every induced subgraph of $K(G)$ is the clique graph of a special clique subgraph of G.*

Proof:

(i) Let H be a special clique subgraph of G and $M_{i_1}, \ldots, M_{i_r}$ be its cliques. Since they are cliques of G, let U be the subgraph of $K(G)$ induced by their corresponding vertices $w_{i_1}, \ldots, w_{i_r}$. Then w_{i_j} is adjacent to $w_{i_{j'}}$ in U if and only if M_{i_j} intersects $M_{i_{j'}}$ in G, if and only if M_{i_j} intersects $M_{i_{j'}}$ in H. So U is isomorphic to $K(H)$.

(ii) Let U be an induced subgraph of $K(G)$ and $w_{i_1}, \ldots, w_{i_s}$ be its vertices. Let $M_{i_1}, \ldots, M_{i_s}$ be the cliques of G that correspond to those vertices.

Consider the subgraph $G_{i_1,\dots,i_s}$. As G is hereditary clique-Helly, by theorem 3.4, G is cliqual. Thus, the cliques of $G_{i_1,\dots,i_s}$ are exactly $M_{i_1},\dots,M_{i_s}$ and $G_{i_1,\dots,i_s}$ is a special clique subgraph. Analogously to the proof of item (i), it follows that U is isomorphic to $K(G_{i_1,\dots,i_s})$. ∎

Theorem 3.6 *Let G be a clique-Helly K-perfect graph. Then G is c–coordinated.*

Proof: Let H be a special clique subgraph of G. As the clique-Helly property is clique-hereditary, H is clique-Helly. By Theorem 3.5, $K(H)$ is an induced subgraph of $K(G)$ and thus, $K(H)$ is perfect. By Theorem 3.1, we conclude that $M(H) = F(H)$. ∎

Remark 3.2 *If G is not a clique-Helly graph but a K-perfect graph, then it still holds that, for every special clique subgraph H of G, $F(H)$ is equal to the maximum number of pairwise intersecting cliques in H.*

Corollary 3.1 *Let G be a perfect clique-Helly graph. Then $K(G)$ is c–coordinated.*

Proof: If G is a clique-Helly graph, $K(G)$ is clique-Helly and $K^2(G)$ is an induced subgraph of G [11]. Then, if G is perfect, $K^2(G)$ is perfect too. So $K(G)$ is K-perfect and clique-Helly, and by Theorem 3.6, $K(G)$ is c–coordinated. ∎

Corollary 3.2 *If G is K-perfect and clique-Helly, then the induced subgraph of G obtained by identifying twin vertices and then removing dominated vertices is c–coordinated.*

Proof: If G is clique-Helly, then $K^2(G)$ is the induced subgraph of G obtained identifying twin vertices and then removing dominated vertices [11]. Corollary 3.1 completes the proof. ∎

Now, we are able to characterize hereditary clique-Helly K-perfect graphs by clique subgraphs.

Theorem 3.7 *Let G be an hereditary clique-Helly graph. Then the following statements are equivalent:*

(i) *G is K-perfect.*

(ii) *G is c–coordinated.*

(iii) *$|C(H)| \leq \alpha_C(H)M(H)$ for every clique subgraph H of G.*

Proof: By Theorem 3.6, item (i) implies item (ii).

(ii) $\Rightarrow$ (i) Let U be an induced subgraph of $K(G)$. By Theorem 3.5, there exists a clique subgraph H of G such that $K(H) = U$. As G is c–coordinated, $F(H) = M(H)$. By hypothesis, G is hereditary clique-Helly and therefore, H is hereditary clique-Helly too. Theorem 3.1 implies that $\chi(U) = \omega(U)$.

(i) $\Rightarrow$ (iii) By the Perfect Graph Theorem [19], a graph G is perfect if and only if, for every induced subgraph H of G, $|V(H)| \leq \alpha(H)\omega(H)$.
Let H be a clique subgraph of G. By Theorem 3.5, $K(H)$ is an induced subgraph of $K(G)$. As $K(G)$ is perfect, $|V(K(H))| \leq \alpha(K(H))\omega(K(H))$. By Theorem 3.1 $\omega(K(H)) = M(H)$, and since $|V(K(H))| = |C(H)|$ and $\alpha(K(H)) = \alpha_C(H)$, it follows that $|C(H)| \leq \alpha_C(H)M(H)$.

(iii) $\Rightarrow$ (i) Let U be an induced subgraph of $K(G)$. By Theorem 3.5, there exists a clique subgraph H of G such that $K(H) = U$. Then, by Theorem 3.1 $\omega(U) = M(H)$, and since $|V(U)| = |C(H)|$ and $\alpha(U) = \alpha_C(H)$, it follows that $|V(U)| \leq \alpha(U)\omega(U)$. Therefore, by the Perfect Graph Theorem [19], $K(G)$ is perfect. ■

4 Clique graphs of clique-Helly perfect graphs

The following theorem due to P.C. Gilmore (see [8]) characterizes clique matrices.

Theorem 4.1 *Let A be a* 0-1 *matrix. Then A is a clique matrix if and only if:*

(i) *A does not have dominated rows.*

(ii) *A does not contain zero columns.*

(iii) *The family of columns of A satisfy the Helly property.*

The classes of clique graphs of clique-Helly and hereditary clique-Helly perfect graphs are analyzed.

Consider the graph $H(G)$ as it is defined in [16], where $V(H(G)) = \{q_1, \ldots, q_k, w_1, \ldots, w_n\}$, each q_i corresponds to the clique M_i of G, and each w_i corresponds to the vertex v_i of G. The edges of $H(G)$ are the following: the vertices $q_1, \ldots, q_k$ induce the graph $K(G)$, the vertices $w_1, \ldots, w_n$ induce a stable set and w_j is adjacent to q_i if and only if v_j belongs to the clique M_i in G.

Let $A \in R^{n\times m}$ and $B \in R^{n\times k}$ be two matrices. We define the matrix $A \mid B \in R^{n\times(m+k)}$ as $(A \mid B)(i,j) = A(i,j)$ for $i = 1, \ldots, n$, $j = 1, \ldots, m$ and $(A \mid B)(i, m+j) = B(i,j)$ for $i = 1, \ldots, n$, $j = 1, \ldots, k$. Let I_n be the $n \times n$ identity matrix.

Theorem 4.2 [16] *Let G be a clique-Helly graph and $H(G)$ as it was defined above. Then the cliques of $H(G)$ are $N[w_i]$ for each i, w_i is a simplicial vertex of $H(G)$ for every i, and $K(H(G)) = G$.*

Corollary 4.1 *If G is a clique-Helly graph such that $|V(G)| = n$, then $A_{H(G)} = A^t_G \mid I_n$.*

Proof: It follows from the previous theorem and the definition of $H(G)$. ■

Theorem 4.3 *H is an injective operator from CH to CH that maps HCH on HCH and $CH \setminus HCH$ on $CH \setminus HCH$.*

Proof: Let G_1 and G_2 be clique-Helly graphs such that $H(G_1) = H(G_2)$. Then, by Theorem 4.2, $G_1 = K(H(G_1)) = K(H(G_2)) = G_2$.

Let G be a clique-Helly graph. Then $A_{H(G)} = A_G^t \mid I_n$ is a clique matrix of $H(G)$. Observe that a family of rows $\{i_1, \ldots, i_s\}$ of $A_{H(G)}$ has a common intersection if and only if the family of columns $\{i_1, \ldots, i_s\}$ of A_G has a common intersection. Since, by Theorem 4.1, the columns of A_G verify the Helly property, then the rows of $A_{H(G)}$ verify the Helly property too. Therefore, $H(G)$ is a clique-Helly graph.

Now, let G be an hereditary clique-Helly graph. Then $A_{H(G)} = A_G^t \mid I_n$ is a clique matrix of $H(G)$. A graph is HCH if and only if its clique matrix does not contain a vertex-edge incidence matrix of a triangle as a submatrix [21]. Suppose that $A_{H(G)}$ contains a vertex-edge incidence matrix B of a triangle as a submatrix. Since B has two 1's by column, it follows that B must be a submatrix of A_G^t, and then B^t is a submatrix of A_G, which is a contradiction, because G is an HCH graph. Hence, $H(G)$ is an HCH graph.

If $G \in CH \backslash HCH$, then A_G contains a vertex-edge incidence matrix of a triangle as a submatrix and, in consequence, one of these matrices is contained by $A_{H(G)}$ too. Therefore, $H(G) \in CH \setminus HCH$. ■

Lemma 4.1 *Let G be a graph and v be a vertex of G such that $N[v]$ induces a complete subgraph in G. Then G is perfect if and only if $G - \{v\}$ is perfect.*

Proof:
$\Rightarrow$) $G - \{v\}$ is an induced subgraph of G, and therefore is perfect.

$\Leftarrow$) Let H be an induced subgraph of G. If v does not belong to H, then H is an induced subgraph of $G - \{v\}$ and therefore $\omega(H) = \chi(H)$. If v belongs to H, $H - \{v\}$ is an induced subgraph of $G - \{v\}$ and then $\omega(H - \{v\}) = \chi(H - \{v\})$. The neighbourhood of v in H is complete, so $|N(v)| \leq \omega(H - \{v\})$. There are two possible cases:

- If $|N(v)| < \omega(H - \{v\})$, then $\omega(H) = \omega(H - \{v\})$ and any optimal coloring of $H - \{v\}$ can be extended to an optimal coloring of H with the same number of colors. Hence, $\chi(H) = \chi(H - \{v\}) = \omega(H - \{v\}) = \omega(H)$.
- If $|N(v)| = \omega(H - \{v\})$, then $\omega(H) = \omega(H - \{v\}) + 1$ and any optimal coloring of $H - \{v\}$ can be extended to an optimal coloring of H giving to v a new color, and so $\chi(H) = \chi(H - \{v\}) + 1 = \omega(H - \{v\}) + 1 = \omega(H)$. ■

Theorem 4.4 *Let G be a graph. G is K-perfect if and only if $H(G)$ is perfect.*

Proof: Let G be a graph and $G_0 = H(G)$ as it was defined, where $V(H(G)) = \{q_1, \ldots, q_k, w_1, \ldots, w_n\}$, the vertices $q_1, \ldots, q_k$ induce the graph $K(G)$, the vertices $w_1, \ldots, w_n$ induce a stable set and w_j is adjacent to q_i if and only if v_j belongs to the clique M_i in G. We define $G_1 = G_0 - \{w_1\}$, $\ldots$, $G_n = G_{n-1} - \{w_n\} = K(G)$. By Theorem 4.2, for every $1 \leq i \leq n$, $N[w_i]$ is complete in G_{i-1}. So by Lemma 4.1, for every $1 \leq i \leq n$, G_i is perfect if and only if G_{i-1} is perfect. Therefore $H(G) = G_0$ is perfect if and only if $G_n = K(G)$ is perfect. ■

Lemma 4.2 *Let $\mathcal{H}$ be a class of graphs and let $\mathcal{L}$ be a class of clique-Helly graphs such that:*

(i) *If G belongs to $\mathcal{H}$ then $K(G)$ belongs to $\mathcal{L}$.*

(ii) *If F belongs to $\mathcal{L}$ then $H(F)$ belongs to $\mathcal{H}$.*

Then $K(\mathcal{H}) = \mathcal{L}$.

Proof: Item (i) implies that $K(\mathcal{H}) \subseteq \mathcal{L}$. On the other hand, let F be a graph in $\mathcal{L}$. By item (ii), $H(F)$ belongs to $\mathcal{H}$. And since F is a clique-Helly graph, by Theorem 4.2 $K(H(F)) = F$. Therefore, $\mathcal{L} \subseteq K(\mathcal{H})$, which completes the proof. ■

Now, we can prove the main results of this section.

Theorem 4.5 *The classes* clique-Helly perfect *and* clique-Helly K-perfect *are dual-clique classes of graphs.*

Proof: Let $\mathcal{H}$ be the class of clique-Helly perfect graphs and $\mathcal{L}$ be the class of clique-Helly K-perfect graphs. Let $G \in \mathcal{H}$. Then $K(G)$ is clique-Helly and $K^2(G)$ is an induced subgraph of G [11], so $K(G) \in \mathcal{L}$. By Theorems 4.4 and 4.3, if $F \in \mathcal{L}$ then $H(F) \in \mathcal{H}$. Therefore, by Lemma 4.2, $K(\mathcal{H}) = \mathcal{L}$.

It follows immediately that if $G \in \mathcal{L}$ then $K(G) \in \mathcal{H}$. By Theorem 4.3, if $F \in \mathcal{H}$ then $H(F)$ is clique-Helly and since $K(H(F)) = F$, it follows that $H(F) \in \mathcal{L}$. Hence, by Lemma 4.2, $K(\mathcal{L}) = \mathcal{H}$. ■

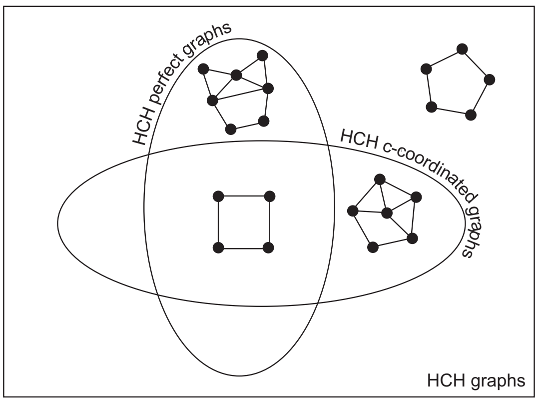

Figure 6: Dual-clique classes HCH perfect and HCH c-coordinated.

Theorem 4.6 *The classes HCH* perfect *and HCH* K-perfect *are dual-clique classes of graphs.*

Proof: It is analogous to the proof of Theorem 4.5 replacing clique-Helly for HCH.

This theorem and the partial characterization of K-perfect graphs (Theorem 3.7) imply the following corollary.

Corollary 4.2 *The classes HCH* perfect *and HCH* c-coordinated *are dual-clique classes of graphs.*

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